Theme: Null controllability of a nonlinear dissipative system and application to the detection of incomplete parameter for a nonlinear population dynamics models

Option: Optimal control

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Let T and A be a positive real, and Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 1$, with smooth boundary $\partial\Omega$.
For given positive real function F we consider the following nonlinear population dynamics model with incomplete data.

$$\begin{cases} \dfrac{\partial y}{\partial t} + \dfrac{\partial y}{\partial a} - \Delta y + \mu y = \xi + \lambda \hat{\xi} \text{ in } Q = (0, T) \times (0, A) \times \Omega \\ y(t, a, x) = 0 \text{ on } \Sigma \\ y(t, 0, x) = F\left(\int_0^A \beta y da \right) \text{ in } (0, T) \times \Omega \\ y(0, a, x) = y^0 + \tau \hat{y}^0 \text{ in } (0, A) \times \Omega, \end{cases} \tag{1}$$

Here $y(t,a,x)$ is a distribution of individuals of age a at time t and location $x \in \Omega$, A is the maximal live expectancy, $\beta(a)$ and $\mu(a)$ denote, respectively, the natural fertility and natural death rate of individuals of age a.
Thus, the formula $\int_0^A \beta(a) y da$ denotes, the distribution of newborn individuals at time t and location x.
For example, in oviparus species it denotes the total eggs at time t and location x. Therefore, the quantity $F(\int_0^A \beta y da)$ is the distribution of eggs that hatches at time t and location x.

For the rest we denote by:

$$Q_\omega = (0,T)\times(0,A)\times\omega \text{ and } Q_\Theta = (0,T)\times(0,A)\times\Theta,$$

where $\overline{\Theta}\subset\Omega$ is the observation domain, and $\overline{\omega}\subset\Omega$ the control domain.
We denote also

$$Q_A = (0,A)\times\Omega \text{ and } Q_T = (0,T)\times\Omega.$$

Null Controllability

For $G \in L^{\infty}(\mathbb{R})$, we consider the following system:

$$\begin{cases} -\dfrac{\partial q}{\partial t} - \dfrac{\partial q}{\partial a} - \Delta q + \mu q = \beta q(t,0,x)\, G\left(\int_0^A \beta q da\right) + \chi_{\Theta} h + \chi_{\omega} v \text{ in } Q \\ q(t,a,x) = 0 \text{ on } \Sigma \\ q(t,A,x) = 0 \text{ in } (0,T)\times(0,A) \\ q(T,a,x) = 0 \text{ in } (0,A)\times\Omega, \end{cases} \tag{2}$$

For this aim we introduce the weight:

$$e_M(t,a) = \exp\left(\frac{M}{at(A-a)(T-t)}\right)$$

and define the Hilbert space at weight:

$$L^2(e_M) = \left\{ f \in L^2(Q) : \int_Q f^2 e_M < \infty \right\}.$$

endowed the natural norm. M is a real that will be defined in the following.

Moreover, we consider the following assumption:

$$(H_1)\begin{cases} \mu(a) \geq 0 \\ \mu \in L^{\infty}(0,A), \end{cases}$$

,

$$(H_2)\begin{cases} \beta \in C^2([0,A]) \\ \beta(a) \geq 0 \text{ in } [0,A] \\ \exists 0 < a_0 < a_1 < A \text{ such that } \beta(a) = 0 \text{ in } (0,a_0) \cup (a_1,A) \end{cases}$$

Theorem

Let $h \in L^2(e_M)$, under the assumptions (H_1) and (H_2) there exists a control $v \in L^2(Q_\omega)$ such that $q(0,a,x) = 0$ a.e in $(0,A) \times \Omega$.

An adapted Observability Inequality

We consider the following system:

$$\begin{cases} \dfrac{\partial w}{\partial t} + \dfrac{\partial w}{\partial a} - \Delta w + \mu w = f \text{ in } Q \\ w = 0 \text{ on } \Sigma \\ w(0, a, x) = w^0 \text{ in } (0, A) \times \Omega \\ w(t, 0, x) = w^1 \text{ in } (0, T) \times \Omega \end{cases} \tag{3}$$

Proposition

There exist positive constants $\alpha_1 > 1$ and $\lambda_1 > 1$ and there exists a positive constants $C(A, T)$ which depends on A and T such that for $\alpha \geq \alpha_1$, $\lambda \geq \lambda_1$ and for all solution of (3) the following inequality holds:

$$\alpha^3\lambda^4 \int_Q e^{-2\alpha\eta} p^3 \left|w^2\right| dxdadt \leq C \int_Q e^{-2\alpha\eta} f^2 dtdadx +$$

$$C\alpha^3\lambda^4 \int_{Q_\omega} e^{-2\alpha\eta} p^3 \left|w^2\right| dxdadt \tag{4}$$

with

$$\eta(t,a,x) = \frac{e^{2\lambda\|\Psi\|_\infty} - e^{\lambda\Psi(x)}}{at(T-t)(A-a)} \text{ and } p(t,a,x) = \frac{e^{\lambda\Psi(x)}}{at(T-t)(A-a)}$$

The proof of the theorem is based on the observability inequality.

Proposition

If $f = 0$, there exists $M > 0$ and $C > 0$ such that the following inequality is true:

$$\int_Q exp\left(\frac{-M}{at(T-t)(A-a)}\right) w^2 dxdadt \leq C \int_{Q_\omega} w^2 dxdadt \tag{5}$$

herein w is the solution of the system (3).

Proof of the theorem

Let $\varphi^0 \in L^2(Q)$ and φ solution of the following system:

$$\begin{cases} \dfrac{\partial\varphi}{\partial t} + \dfrac{\partial\varphi}{\partial a} - \Delta\varphi + \mu\varphi = 0 \text{ in } Q \\ \varphi = 0 \text{ on } \Sigma \\ \varphi(0,a,x) = \varphi^0 \text{ in } (0,A)\times\Omega \\ \varphi(t,0,x) = \int\limits_0^A \beta\varphi G(\eta)\, da \text{ in } (0,T)\times\Omega. \end{cases} \tag{6}$$

For $\delta > 0$ we consider the functional J_δ define by:

$$K_\delta(\varphi^0) = \frac{1}{2}\int_{Q_\omega} \varphi^2 dxdadt + \int_{Q_\Theta} h\varphi dxdadt + \delta \left\|\varphi^0\right\|_{L^2(Q_A)} \tag{7}$$

The functional J_δ is continuous, strictly convex and coercive, then, there exist $\hat{\varphi}^0 \in L^2((0,A)\times\Omega)$ such that

$$K_\delta\left(\hat{\varphi}^0\right) = min_{\varphi^0\in L^2((0,A)\times\Omega)} K_\delta(\varphi^0).$$

If $\widehat{\varphi^0} \neq 0$, using the observability inequality, we prove that, there exists $C > 0$ independently of δ such that

$$\int_{Q_\omega} |\hat{\varphi}|^2 \, dxdadt \leq C\,\|h\|^2_{L^2(e_M)} \tag{8}$$

We consider now the following system:

$$\begin{cases} -\frac{\partial q_\delta}{\partial t} - \frac{\partial q_\delta}{\partial a} - \Delta q_\delta + \mu q_\delta = \beta q_\delta(t,0,x)\, G(\eta) + \chi_\Theta h + \chi_\omega \hat{\varphi} \text{ in } Q \\ q_\delta(t,a,x) = 0 \text{ on } \Sigma \\ q_\delta(t,A,x) = 0 \text{ in } (0,T)\times(0,A) \\ q_\delta(T,a,x) = 0 \text{ in } (0,A)\times\Omega, \end{cases} . \qquad (9)$$

Using the optimality condition of (7) and the inequality (8) if $\hat{\varphi}^0 \neq 0$, and $lim_{\lambda \longrightarrow 0} \frac{K_\delta(\lambda\varphi^0)}{\lambda} \geq 0$ if $\hat{\varphi}^0 = 0$, we prove that there exist $K_0 > 0$ such that q_δ verify:

$$\int_0^A \int_\Omega q_\delta^2(0,a,x)\, dxda \leq \delta K_0. \qquad (10)$$

Fixed point:

Let Λ be an operator defined by

$$\eta \in L^2(Q_T) \longmapsto \Lambda(\eta) = \int_0^A \beta q_\delta(\eta) da$$

where $q_\delta(\eta)$ is the solution of the system (9)

By asking $Y(\eta) = \int_0^A \beta q_\delta(\eta) da$, we prove that $Y(\eta)$ is the solution of the following system:

$$\begin{cases} \partial_t Y(\eta) - \Delta Y(\eta) - \int_0^A \beta q_\delta(\eta) da = Z(\eta) \text{ in } Q_T \\ Y(\eta)(t,x) = 0 \text{ on } (0,T) \times \partial\Omega \\ Y(\eta)(0,x) = 0 \text{ in } \Omega \end{cases} \tag{11}$$

where $Z(\eta) = \int_0^A (\beta' q_\delta(\eta) + \beta \hat{\varphi}_\delta \chi_\omega + \beta^2 q_\delta(\eta)(t,0,x) G(\eta) + \beta h) da$.

We prove, using the system (11), that the operator Λ is continuous, bounded and compact in $L^2((0,T)\times\Omega)$. By Schauder's fixed point, Λ admitt a fixed point and we obtain the approached controllability of the nonlinear system. As the control $\hat{\varphi}_\delta$ is bounded independently of δ and

$$\int_0^A \int_\Omega q_\delta^2(0,a,x)\,dxda \leq \delta K_0, \tag{12}$$

we obtain the null controllability.

Sentinel method

For given positive real function F we consider the following nonlinear population dynamics model.

$$\begin{cases} \dfrac{\partial y}{\partial t} + \dfrac{\partial y}{\partial a} - \Delta y + \mu y = \xi + \lambda \hat{\xi} \text{ in } Q = (0,T) \times (0,A) \times \Omega \\ y(t,a,x) = 0 \text{ on } \Sigma \\ y(t,0,x) = F\left(\int\limits_0^A \beta y da \right) \text{ in } (0,T) \times \Omega \\ y(0,a,x) = y^0 + \tau \hat{y}^0 \text{ in } (0,A) \times \Omega, \end{cases} \tag{13}$$

The question is as follow:

$$\left\{ \begin{array}{c} \text{Can we get from the observation } m_0, \text{ the informations on } \lambda\hat{\xi} \\ \text{which are independent of the variations of initial condition around } y^0? \end{array} \right. \tag{14}$$

We ask $y_{obs} = m_0$, the observation of the system, that is a function in $L^2\left(Q_\Theta\right)$.

For the sequel we assume that the following assumption hold:

$$(H_3): F \in C^1(\mathbb{R}) \text{ and } F' \in L^\infty(\mathbb{R}).$$

Under the assumptions (H_1) and (H_2) the system (13) admit a unique solution in

$$W((0,T)\times(0,A);H^1(\Omega)) = \{y \in L^2\left((0,T)\times(0,A);H^1(\Omega)\right) / \partial_t y + \partial_a y \in L^2\left((0,T)\times(0,A);H^{-1}(\Omega)\right)\}$$

that we design by $y(\lambda,\tau) = y(t,a,x,\lambda,\tau)$.

We have this result:

proposition

The functions $\lambda \mapsto y(\lambda,\tau)$ and $\tau \mapsto y(\lambda,\tau)$ are differentiable at the point 0.

Moreover, we have $Y_\tau = lim_{\tau \longrightarrow 0} \dfrac{y(\lambda,\tau) - y(\lambda,0)}{\tau}$ solution of this following system:

$$\begin{cases} \dfrac{\partial Y_\tau}{\partial t} + \dfrac{\partial Y_\tau}{\partial a} - \Delta Y_\tau + \mu Y_\tau = 0 \text{ in } Q \\ Y_\tau(t,a,x) = 0 \text{ on } \Sigma \\ Y_\tau(t,0,x) = \int\limits_0^A \left(\beta F' \left(\int\limits_0^A \beta y(\lambda,0) da \right) Y_\tau \right) da \text{ in } (0,T) \times \Omega \\ Y_\tau(0,a,x) = \hat{y}^0 \text{ in } (0,A) \times \Omega. \end{cases} \tag{15}$$

Proof.

Let us consider $\hat{y}(\tau) = \dfrac{e^{-\lambda_0 t}(y(\lambda,\tau) - y(\lambda,0))}{\tau}$ with $\lambda_0 > 0$

and z solution of this system

$$\begin{cases} \dfrac{\partial z}{\partial t} + \dfrac{\partial z}{\partial a} - \Delta z + (\mu + \lambda_0) z = 0 \text{ in } Q \\ z(t,a,x) = 0 \text{ on } \Sigma \\ z(t,0,x) = \int\limits_0^A \left(\beta F' \left(\int\limits_0^A \beta y(\lambda,0) da \right) e^{-\lambda_0 t} z \right) da \text{ in } (0,T) \times \Omega \\ z(0,a,x) = \hat{y}^0 \text{ in } (0,A) \times \Omega. \end{cases} \tag{16}$$

We prove that $lim_{\tau \longrightarrow 0}(\hat{y}(\tau) - z) = 0 a.e$ in $L^2((0,T) \times (0,A); H^1(\Omega))$ □

Now let $h \in L^2(Q_\Theta)$ such that

$$\int_{Q_\Theta} hdxdadt = 1;$$

For a control function $v \in L^2(Q_\omega)$, we define the functional:

$$s(\lambda,\tau) = \int_{Q_\Theta} hy(\lambda,\tau)\, dxdadt + \int_{Q_\omega} vy(\lambda,\tau)\, dxdadt, \tag{17}$$

we have:

$$\frac{\partial S}{\partial \tau}(0,0) = \int_{Q_\Theta} hy_\tau dxdadt + \int_{Q_\omega} vy_\tau dxdadt. \tag{18}$$

where $y_\tau = lim_{\tau \longrightarrow 0} \dfrac{y(0,\tau) - y(0,0)}{\tau}$

We say that S defines a sentinel for the problem (13) if there exists v such that S is insensitive (at first order) with respect to the missing term $\tau\hat{y}^0$, this means

$$\frac{\partial S}{\partial \tau}(0,0) = 0, \tag{19}$$

for any $\hat{y}^0$ and if the norm $\|v\|_{L^2(Q_\omega)}$ of v is minimal.

Reduction of the problem

The existence of sentinel is equivalent to a controllability problem.

Let us consider the following system:

$$\begin{cases} -\frac{\partial q}{\partial t} - \frac{\partial q}{\partial a} - \Delta q + \mu q = \beta q(t,0,x) F'\left(\int_0^A \beta y(0,0) da\right) + \chi_\Theta h + \chi_\omega v \text{ in } Q \\ q(t,a,x) = 0 \text{ on } \Sigma \\ q(t,A,x) = 0 \text{ in } (0,T)\times\Omega \\ q(T,a,x) = 0 \text{ in } (0,A)\times\Omega. \end{cases} \tag{20}$$

where $y(0,0)$ is the solution of system (13) with $\lambda = \tau = 0$.

Proposition

The sentinel is equivalent to finding $v \in L^2\left(Q_\omega\right)$ of a minimal norm such that

$$q\left(0, a, x\right) = 0 \text{ a.e in } \left(0, A\right) \times \Omega.$$

The function

$$y_\lambda = \lim_{\lambda \to 0} \frac{y(\lambda, 0) - y(0,0)}{\lambda}$$

is the solution of the system

$$\begin{cases} \dfrac{\partial y_\lambda}{\partial t} + \dfrac{\partial y_\lambda}{\partial a} - \Delta y_\lambda + \mu y_\lambda = \hat{\xi} \text{ in } Q \\ y_\lambda(t,a,x) = 0 \text{ on } \Sigma \\ y_\lambda(t,0,x) = \int_0^A \left(\beta F' \left(\int_0^A \beta y(0,0) da \right) y_\lambda \right) da \text{ in } (0,T) \times \Omega \\ y_\lambda(0,a,x) = 0 \text{ in } (0,A) \times \Omega \end{cases} \tag{21}$$

We suppose that $\omega \subset \Theta$.

We have:

$$S_{obs}(\lambda,\tau) \simeq S(0,0) + \lambda \frac{\partial S}{\partial \lambda}(0,0) + \tau \frac{\partial S}{\partial \tau}(0,0) \tag{22}$$

equivalent

$$S_{obs}(\lambda,\tau) - S(0,0) \simeq \lambda \frac{\partial S}{\partial \lambda}(0,0). \tag{23}$$

Then

$$\int_Q (\chi_\Theta h + \chi_\omega v)(m_0 - y(0,0))\,dxdadt = \lambda \int_Q (\chi_\Theta h + \chi_\omega v) y_\lambda \,dxdadt \tag{24}$$

Multiplying the first equation of (20) by y_λ and integrating by parts, we obtain

$$\int_Q q\hat{\xi}dxdadt \simeq \int_Q (\chi_\Theta h + \chi_\omega v)\, y_\lambda dxdadt \tag{25}$$

and then

$$\int_Q \lambda\hat{\xi}qdxdadt \simeq \int_Q (\chi_\Theta h + \chi_\omega v)\,(m_0 - y(0,0))\, dxdadt. \tag{26}$$

Knowing h, v, m_0 and $y(0,0)$, q can be calculated. Thus we obtain an integral equation in $\lambda\hat{\xi}$.

Thanks for your kind Attention !!!!!