Theme:

Controllability in Population Dynamics with age, size Structuring and Diffusion

Option: Optimal control

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Introduction

This talk is most devoted to gives the results of null controllability in polulation dynamics structuring with age size and spatial diffusion.

We therefore consider the following population dynamics model:

$$\begin{cases} \dfrac{\partial y}{\partial t} + \dfrac{\partial y}{\partial a} + \dfrac{\partial y}{\partial s} - \Delta y + \mu(a) y = 0 & \text{in } Q, \\ \dfrac{\partial y}{\partial \vartheta} = 0 & \text{on } \Sigma, \\ y(x,0,s,t) = \int_0^A \int_0^S \beta(a,\hat{s},s) y(x,a,\hat{s},t)\, da d\hat{s} & \text{in } \Omega \times (0,S) \times (0,T) \\ y(x,a,s,0) = y_0(x,a,s) & \text{in } \Omega \times (0,A) \times (0,S); \\ y(x,a,0,t) = 0 & \text{in } \Omega \times (0,A) \times (0,S). \end{cases} \quad (1)$$

Here $y(x,a,s,t)$ is a distribution of individuals of age a, size s at time t and location $x \in \Omega$, A and S are respectively the maximal live expectancy and the maximal size, $\beta(a,\hat{s},s)$ and $\mu(a)$ denote, respectively, the natural fertility and natural death rate of individuals.

- $\dfrac{\partial y}{\partial a}$ is the aging,
- $\dfrac{\partial(g(s)y)}{\partial s}$ is the growth. Here $g(s) = 1$
- Δy the diffusion,
- $-\mu(a)y$ the mortality.

Thus, the formula $\int_0^A \int_0^S \beta(a,\hat{s},s)y(x,a,\hat{s},t)dad\hat{s}$ denotes, the distribution of newborn individuals of size s at time t and location x.

In this models size is viewed as a continuum variables specific to individuals, such as mass, volume, length, maturity, bacterial or viral load, or other physiologic or demographic property. It is assumed that size increases in the same way for all individuals in the population, as controlled by a growth modulus g(s).

For the sequel we assume that the following assumption hold:

$$(H_1) = \begin{cases} \mu(a) \geq 0 \text{ for every } a \in (0, A) \\ \mu \in L^1[0, A^*] \quad A^* \in (0, A) \\ \int_0^A \mu(a) da = +\infty \end{cases} .$$

,

$$(H_2) = \begin{cases} \beta(a, \hat{s}, s) \in C\left([0, A] \times [0, S]^2\right) \\ \beta(a, \hat{s}, s) \geq 0 \text{ in } \left([0, A] \times [0, S]^2\right) \end{cases} ,$$

And we consider the following hypotheses

$$(H_3): \ \beta(a, \hat{s}, s) = 0 \ \forall a \in (0, \hat{a}) \text{ where } \hat{a} \in (0, A).$$

$$(H_4): \ \beta(a, \hat{s}, s) = 0 \ \forall s \in (s_e, S) \text{ where } s_e \in (0, S).$$

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Semi-group, Observability and Null Controllability

We denote by $K = L^2(\Omega \times (0,A) \times (0,S))$ the state space of the system and we define the population (age-size-space diffusion) operator with diffusion: $\mathcal{A} : D(\mathcal{A}) \longrightarrow K$ as follows:

$$\mathcal{A}\varphi = -\frac{\partial \varphi}{\partial a} - \frac{\partial \varphi}{\partial s} - \mu(a)\varphi + \Delta\varphi, \quad \forall \varphi \in D(\mathcal{A}) \tag{2}$$

where

$$D(\mathcal{A}) =$$

$$\{\varphi / \mathcal{A}\varphi \in K,\ \varphi(a,0,s) = 0,\ \frac{\partial \varphi}{\partial \nu}|_{\partial\Omega} = 0, \varphi(x,0,s) = \int_0^A \int_0^S \beta(a,\hat{s},s)\varphi(x,a,\hat{s})\,da\,d\hat{s}\}$$

Proceeding as in the work of W. L. Chen and Bao-Zhu Guo: On the semigroups of age-size dependent population dynamics with spatial diffusion, we showed this result:

Semigroup

Under the assumption $(H1)$ and $(H2)$ the operator $(\mathcal{A}, D(\mathcal{A}))$ is the infinitesimal generator of a strongly continuous semigroup $\mathbb{U}$ on K.

From there we can define the adjoint operator by:

$$for \quad all \quad \psi \in D(\mathcal{A}^*)$$

$$\mathcal{A}^*\psi = \partial_a\psi + \partial_s\psi - \mu(a)\psi + \Delta\psi + \int_0^S \beta(a,s,\hat{s})\psi(x,a,\hat{s})d\hat{s}$$

We also introduce the input space $U \subset K$ and the control operator $\mathcal{B} \in \mathcal{L}(U, K)$ defined by

$$\mathcal{B}u = mu \ (u \in U). \tag{3}$$

With above notation, we rewrite the system (1) by:

$$\begin{cases} \dot{y} = \mathcal{A}y + \mathcal{B}u(t) \\ y(0) = y_0. \end{cases} \tag{4}$$

As mentioned above, the null-controllability of a pair $(\mathcal{A}, \mathcal{B})$ is equivalent to the final state observability of the pair $(\mathcal{A}^*, \mathcal{B}^*)$.

Recall that the final-state observability of $(\mathcal{A}^*, \mathcal{B}^*)$ is defined as

Definition

The pair $(\mathcal{A}^*, \mathcal{B}^*)$ is final observable in time T if there exists a $K_T > 0$ such that

$$\int_0^T \|\mathcal{B}^* \mathbb{U}_t^* q_0\|^2 \geq K_T^2 \|\mathbb{U}_T^* q_0\|^2 \ (q_0 \in D(\mathcal{A}^*)). \tag{5}$$

The first main result

$$
\begin{cases}
\dfrac{\partial y}{\partial t} + \dfrac{\partial y}{\partial a} + \dfrac{\partial y}{\partial s} - \Delta y + \mu(a) y = \chi_{\Theta_1} u & \text{in } Q, \\
\dfrac{\partial y}{\partial \vartheta} = 0 & \text{on } \Sigma, \\
y(x, 0, s, t) = \int_0^A \int_0^S \beta(a, \hat{s}, s) y(x, a, \hat{s}, t) da d\hat{s} & \text{in } \Omega \times (0, S) \times (0, T) \\
y(x, a, s, 0) = y_0(x, a, s) & \text{in } \Omega \times (0, A) \times (0, S); \\
y(x, a, 0, t) = 0 & \text{in } \Omega \times (0, A) \times (0, S).
\end{cases}
\tag{6}
$$

Here :

- $u(x, a, s, t)$ is the control;
- $\Theta_1 = \omega \times (a_1, a_2) \times (s_1, s_2) \subset \Omega \times (0, A) \times (0, S)$ is the support of the control
- The goal is to drive the solution of the system at a given final times $T > 0$ at zero.

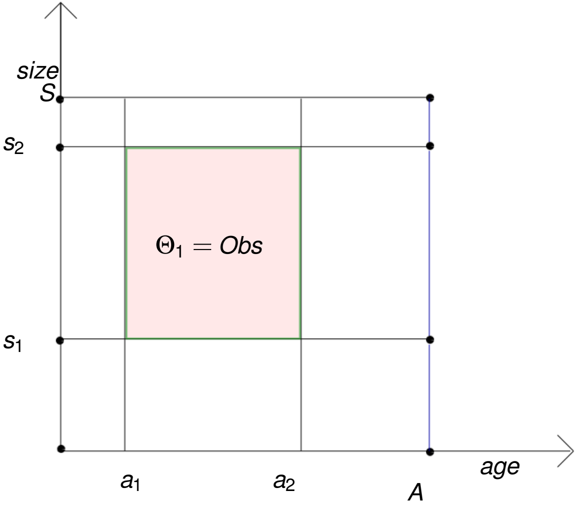


Figure: Here is the support Θ_1 without the space variable

The corresponding adjoint system is given by:

$$\begin{cases} \dfrac{\partial q}{\partial t} - \dfrac{\partial q}{\partial a} - \dfrac{\partial q}{\partial s} - \Delta q + \mu(a) q = \int_0^S \beta(a,s,\hat{s}) q(x,0,\hat{s},t) da d\hat{s} \quad \text{in } Q, \\ \dfrac{\partial q}{\partial \vartheta} = 0 \text{ on } \Sigma, \\ q(x,A,s,t) = 0 \text{ in } \Omega \times (0,S) \times (0,T) \\ q(x,a,s,0) = q_0(x,a,s) \text{ in } \Omega \times (0,A) \times (0,S); \\ q(x,a,S,t) = 0 \text{ in } \Omega \times (0,A) \times (0,S). \end{cases} \tag{7}$$

The question is whether

$$\int_0^S \int_0^A \int_\Omega q^2(x,a,s,T) dx da ds \le C_T \int_0^T \int_{\Theta_1} q^2(x,a,s,t) dx da ds dt. \tag{8}$$

the observation being made in subset

$$\Theta_1 = \omega \times (a_1, a_2) \times (s_1, s_2) \subset \Omega \times (0,A) \times (0,S).$$

The answer is affirmative provided that β satisfies (H_3), $T_0 < \min\{a_2, \hat{a}\} - a_1$, and $T > A - a_2 + a_1 + 2T_0$, where $T_0 = \sup\{s_1, S - s_2\}$.

Theorem

Assume that β and μ satisfy the conditions $(H1) - (H2)$ above.

Assume that $a_1 < \hat{a}$ and the assumption $(H3)$ hold. If $T_0 + a_1 < \min\{a_2, \hat{a}\}$, the pair $(\mathcal{A}^*, \mathcal{B}^*)$ is final-state observable for every $T > A - a_2 + a_1 + 2T_0$. In other words, for every $T > A - a_2 + a_1 + 2T_0$, there exist $K_T > 0$ such that the solution q of (7) satisfies

$$\int_0^S \int_0^A \int_\Omega q^2(x, a, s, T)dxdads \leq K_T \int_0^T \int_{\Theta_1} q^2(x, a, s, t)dxdadsdt.$$

Proof of the theorem

For the proof we adapt the results of Debayan, Tucsnak and Enrique, in "Controllability of a Class of Infinite Dimensional Systems with Age Structure".

Using the semigroup, the caracteristics method and Duhamel Formula, the solution of the adjoint system is given by:

$$q(t) = \begin{cases} q_0(., a+t, s+t)e^{tL} + \\ \int_0^t \left(e^{(t-l)L} \int_0^{s_e} \beta(a+t-l, s+t-l, \hat{s}) q(x, 0, \hat{s}, l) d\hat{s} \right) dl \text{ in } A_1, \\ \int_{\sup\{t-A+a, t-S+s\}}^t \left(e^{(t-l)L} \int_0^S \beta(a+t-l, s+t-l, \hat{s}) q(x, 0, \hat{s}, l) d\hat{s} \right) dl \text{ in } A_2; \end{cases} \tag{9}$$

with $L\psi = \Delta\psi - \mu(a)\psi$

and where

$$A_1 = \{(x,a,s,t) \in Q \text{ such that } 0 < t < A-a \text{ and } 0 < t < S-s\},$$

$$A_1' = \{(x,a,s,t) \in Q \text{ such that } S-s > t > A-a \text{ or } t > S-s > A-a\},$$

$$A_2' = \{(x,a,s,t) \in Q \text{ such that } A-a > t > S-s \text{ or } t > A-a > S-s\}.$$

We denote by $A_2 = A_1' \cup A_2'$. Then $Q = A_1 \cup A_2$.

Proof of the theorem

Here, we need to estimate the nonlocal term

$$\int_0^S \beta(a,s,\hat{s})q(x,0,\hat{s},t)d\hat{s}. \quad (10)$$

For, $\beta = 0$ in $(0,\hat{a})$, we obtain for $T > a_1 + T_0$, $\eta > a_1 + T_0$ and $a_1 < a_0 < \min\{\hat{a}, a_2\} - T_0$ the following estimations:

$$\int_0^S \int_0^{a_0} \int_\Omega q^2(x,a,s,T)dxdads \leq C_T \int_0^T \int_{\Theta_1} q^2(x,a,s,t)dxdsdadt. \quad (11)$$

and

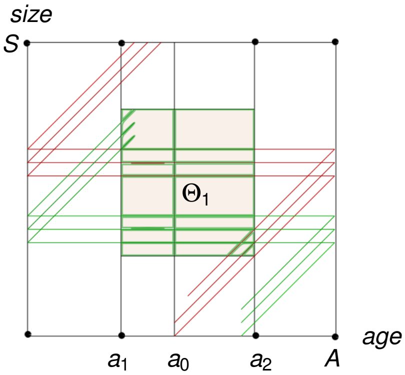
$$\int_\eta^T \int_0^S \int_\Omega q^2(x,0,s,t)dxdsdt \leq C \int_0^T \int_{\Theta_1} q^2(x,a,s,t)dxdsdt. \quad (12)$$

For $T > A - a_2 + a_1 + 2T_0$ and $a_1 + T_0 < \min\{\hat{a}, a_2\}$, there exists a_0 verifying $a_1 < a_0 < \min\{\hat{a}, a_2\} - T_0 \quad and \quad \eta > T_0 + a_1$, such that we had:

$$\int_0^S \int_0^A \int_\Omega q^2(x,a,s,T)dxdads \leq C \int_0^T \int_0^S \int_0^{a_0} \int_\Omega q^2(x,a,s,T)dxdads$$

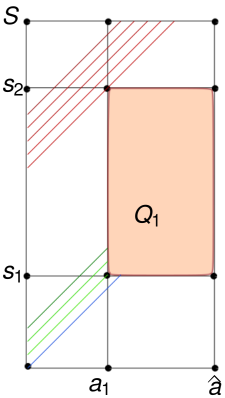
$$+ \int_\eta^T \int_0^S \int_\Omega q^2(x,0,s,t)dxdsdt.$$

from (9).



Here $a_0 = a_2 - T_0$

Figure: Illustration of Observability



$$T_1 = \sup\{s_1, a_1 + S - s_2\}$$

Figure: Here we have chosen $a_2 = \hat{a}$. Since $t > T_1$ all the backward characteristics starting from $(0, s, t)$ enters the observation domain (the green and blue lines), or without the domain by the boundary $s = S$ (red line).

The second main result

$$\begin{cases} \dfrac{\partial y}{\partial t}+\dfrac{\partial y}{\partial a}+\dfrac{\partial y}{\partial s}-\Delta y+\mu(a)y=u(x,a,s,t)\chi_{\Theta_2} & \text{in } Q, \\ \dfrac{\partial y}{\partial \vartheta}=0 & \text{on } \Sigma, \\ y(x,0,s,t)=\int_0^A\int_0^S \beta(a,\hat{s},s)y(x,a,\hat{s},t)da d\hat{s} & \text{in } \Omega\times(0,S)\times(0,T) \\ y(x,a,s,0)=y_0(x,a,s) & \text{in } \Omega\times(0,A)\times(0,S); \\ y(x,a,0,t)=0 & \text{in } \Omega\times(0,A)\times(0,S). \end{cases} \tag{13}$$

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- $u(x,a,s,t)$ is the control;
- $\Theta_2=\{(x,a,s)\in\Omega\times(0,A)\times(0,S)/\quad x\in\omega\quad a\in(a_1,a_2)\quad a-a_0\leq s\leq a+s_e\}$

is the support of the control

- The goal is to drive the solution at a given final times $T>0$ at zero.

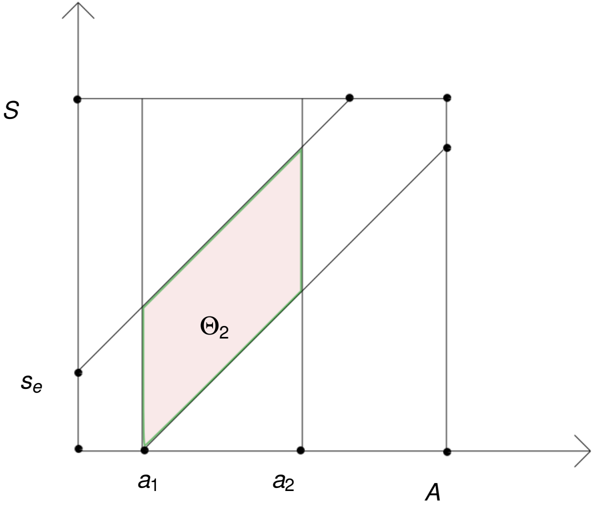

Figure: Here is the support Θ_2 without the space variable

The question is whether

$$\int_0^S \int_0^A \int_\Omega q^2(x,a,s,T)\,dxdads \leq C_T \int_0^T \int_{\Theta_2} q^2(x,a,s,t)\,dxdsdadt. \tag{14}$$

the observation being made in subset

$$\Theta_2 = \{(x,a,s) \in \Omega \times (0,A) \times (0,S) / \quad x \in \omega \quad a \in (a_1,a_2) \quad a - a_0 \leq s \leq a + s_e\}.$$

The answer is affirmative if the hypotheses $(H3)$ and $(H4)$ hold.

This condition means that newborns are smaller than s_e.

Theorem

Assume that β *and* μ *satisfy the conditions* $(H1)-(H2)$ *above. Assume that the hypotheses* $(H3)$ *and* $(H4)$ *above and the condition* $a_0 \in [a_1, \hat{a}]$ *hold. Then the* $(\mathcal{A}^*, B^*)$ *is final-state observable for every* $T > \sup\{a_1 + A - a_0, S - s_e\}$*. In other words, for every* $T > \sup\{a_1 + A - a_0, S - s_e\}$*, there exist* $K_T > 0$ *such that the solution* q *of* (7) *satisfies*

$$\int_0^S \int_0^A \int_\Omega q^2(x,a,s,T)dxdads \leq K_T \int_0^T \int_{\Theta_2} q^2(x,a,s,t)dxdadsdt.$$

Thanks for your kind
Attention !!!!!