1. Introduction

Today's scientists, to better control and exploit their environment, tend to model certain physical processes through mathematical equations. One of the stages of this modeling is the creation of an effective control on the phenomena addressed: this is the control theory. This makes it possible to bring a system from a given initial state to a desirable final state while respecting certain criteria. A control system is a dynamic system that can be acted upon by means of a command. It is generally designed with the aim of improving our quality of life and facilitating certain tasks or even determining optimal solutions for a certain optimization criterion (optimal control).

We analyze the controllability, in infinite dimension, of certain problems modeled by partial differential equations, notably internal control, edge control and stabilization. Currently our work focuses on the control of sediment transport in shallow waters. In this phenomenon the flow of water is modeled by the Saint-Venant equations and the evolution of the bottom is modeled by the Exner equation. The overall objective is to develop a mathematical model adapted to the transport of sediments in the basins and to establish an optimal control system for this transport in order to clean up these basins.

Our current contribution is at two levels. In order to be able to quantify and predict the transport of sediments in hydraulic structures such as dams, dissipation basins and rivers as well as lakes, we have developed a Godunov type diagram capturing all the stationary states. Furthermore, given the importance of water stability in the field of irrigation and in navigable rivers, we analyzed the flow of water in a trapezoidal canal. This analysis made it possible to arrive at a choice of explicit boundary conditions making it possible to guarantee the stability of the water flow.

The most general version of $1D$ Saint-Venant equations with arbitrary varying slope, section profile and friction model is given by the following system (see Adhémar Barré, 1871):

$$\begin{cases} \partial_t A + \partial_x (AV) = 0, \\ \partial_t (AV) + \partial_x (V^2 A) + gA\partial_x H = gA\big(S_0 - S_f(A, V, x)\big), \end{cases} \tag{1.1}$$

section, V is the velocity, $H(A, x)$ is the water depth, g is the gravity acceleration, S_0 is the channel slope, S_f is the friction slope, it is usually defined by semi-empirical formulae proposed by hydraulic engineers in the late nineteenth or early twentieth centuries. In general, it only depends on the fluid quantities. One of the most popular is $S_f = k\dfrac{V^2}{A}$, where k is a constant friction coefficient.

In the trapezoidal chennel, the system (1.1) can now be written as
Dans un canal rectangulaire elles peuvent etre simplifier pour donner

$$\begin{cases} \partial_t H + \partial_x (HU) = 0, \qquad x \in \mathbb{R}, \quad t > 0 \\ \partial_t HU + \partial_x \big(HU^2 + g\dfrac{H^2}{2}\big) = -gH\big(S_0 - S_f(H, V, x)\big) \end{cases} \tag{1.2}$$

To take into account the transport of sediment we introduce the Exner equation given by:

$$(1 - \phi)\partial_t B + \partial_x Q_s = 0, \tag{1.3}$$

where $B(x)$ defines the bottom topography, ϕ is the porosity of the sediment layer ϕ which will be set to zero hereafter and Q_s is the sediment flow. It is fundamental ingredients of the model since they ensure the coupling between the fluid and the solid parts. It is defined by semi-empirical formulas.

2. Godunov-type schemes for the Saint-Venant-Exner model

The present work is devoted to the numerical approximation of the solutions of the nonlinear 1D shallow water model coupled with a sediment continuity equation (Exner Equation) given by the Saint-Venant-Exner system. This system is widely used to model bedload sediment transport phenomena that occur in large time and space scales in river hydraulics or coastal studies. It takes the form of a system of three equations where the first two ones are nothing but the shallow water equations with topography and friction source terms whereas the last equation is a simple conservation law that refers to the evolution in time of the topography due to the action of the fluid. He gave as follows :

$$\begin{cases} \partial_t H + \partial_x (HU) = 0, \qquad x \in \mathbb{R}, \quad t > 0 \\ \partial_t HU + \partial_x \big(HU^2 + g\dfrac{H^2}{2}\big) = -gH\big(S_0 - S_f(H,V,x)\big) \\ (1-\phi)\partial_t B + \partial_x Q_s = 0 \end{cases} \tag{2.1}$$

with

$$S_0(H,b) = \begin{pmatrix} 0 \\ -gH\partial_x b \\ 0 \end{pmatrix} \tag{2.2}$$

For the sake of simplicity in future notations, we have defined

$$q = UV, \qquad , w = \begin{pmatrix} H \\ q \end{pmatrix}, \quad f(w,b) = \begin{pmatrix} q \\ \dfrac{q^2}{H} + \dfrac{gH^2}{2} \\ q_s \end{pmatrix} \tag{2.3}$$

Thus the vector formulation of (2.1) is defined by

$$\partial_t (w,b)^T + \partial_x f(w,b) = S_0(w,b) + S_f(w,b). \tag{2.4}$$

We introduce a Godunov finite volume scheme for (2.1) when the friction term is neglected,

$$\partial_t (w,b)^T + \partial_x f(w,b) = S_0(w,b). \tag{2.5}$$

Consider a uniform mesh defined by the sequence of points $(x_{i+1/2})_{i\in\mathbb{Z}}$ such as, $x_{i+1/2} = x_{i-1/2} + \Delta x$, with Δx the step in assumed constant space. Thus, the space is divided into cells $C_i = (x_{i-1/2}, x_{i+1/2})$ centered in $x_i = x_{i-1/2} + \Delta x/2$.
In the same way, we define the sequence $(t^n)_{n\in\mathbb{N}}$ par $t^0 = 0$ and $t^{n+1} = t^n + \Delta t$, where Δt is the time step that must be restricted by a CFL condition.
We also adopt the ratings:

$$Y_L = \frac{1}{\Delta x}\int_{-\Delta x}^{0} Y(x)dx, \quad Y_R = \frac{1}{\Delta x}\int_{0}^{\Delta x} Y(x)dx \quad \text{and } Y_i = \frac{1}{\Delta x}\int_{C_i} Y(x)dx. \quad \forall Y \in \{h_0, q_0, b_0\}$$

At the date t^n, it is assumed to be known a constant approximation by pieces of the solution of (2.5) and defined as follows:

$$W^n_\Delta(x,t^n) = (w_i^n, b_i^n)^T \quad \text{si} \quad x \in C_i.$$

We are now looking for an updated approximation (w_i^{n+1}, b_i^{n+1}) of the solution at time $t^{n+1} = t^n + \Delta t$. For this we will solve the following problem:

$$\begin{cases} \partial_t (w,b)^T + \partial_x f(w,b) = S_0(w,b) \\ \big(w^n(x,t^n), b^n(x,t^n)\big)^T = W^n_\Delta(x,t^n) \end{cases}$$

Locally, on each interface $x_{i+1/2}$, we are therefore led to consider a Riemann problem between two constant states associated with cells C_i and C_{i+1} defined by:

$$\begin{cases} \partial_t (w,b)^T + \partial_x f(w,b) = S_0(w,b) \\ \big(w^n(x,t^n), b^n(x,t^n)\big)^T = \begin{cases} (w_i^n, b_i^n)^T & \text{si} \quad x<0 \\ \\ (w_i^{n+1}, b_i^{n+1})^T & \text{si} \quad x>0 \end{cases} \end{cases} \tag{2.6}$$

Using the approximate consistency condition (3.4) and the (??)-(??) equities, we obtain

$$(W_i^{n+1}, b_i^{n+1})^T = (w_i^n, b_i^n)^T - \frac{\Delta t}{\Delta x}\big(F(w_i^n, w_{i+1}^n, b_i^n, b_{i+1}^n) - F(w_{i-1}^n, w_i^n, b_{i-1}^n, b_i^n)\big)$$
$$+ \frac{\Delta t}{2}(\bar{S}_0(w_i^n, w_{i+1}^n, b_i, b_{i+1}) + \bar{S}_0(w_{i-1}^n, w_i^n, b_{i-1}, b_i)) \tag{2.7}$$

with the given digital stream function

$$F(w_L, w_R, b_L, b_R) = \frac{1}{2}\big(f(w_R,b_R) + f(w_L,b_L)\big) - \frac{\Delta x}{4\Delta t}((w_R,b_R)^T - (w_L,b_L)^T) + \frac{\Delta x}{2\Delta t}(J_{LR}^+ - J_{LR}^-), \tag{2.8}$$

where

$$J_{LR}^- = \frac{1}{\Delta x}\int_{-\Delta x/2}^{0} \tilde{W}_{\mathcal{R}}(\frac{x}{\Delta t} w_L, w_R, b_L, b_R)dx$$
$$J_{LR}^+ = \frac{1}{\Delta x}\int_{0}^{\Delta x/2} \tilde{W}_{\mathcal{R}}(\frac{x}{\Delta t} w_L, w_R, b_L, b_R)dx.$$

where $\tilde{W}_{\mathcal{R}}(x/t; w_L, w_R, b_L, b_R)$ is the approximate Riemann solver and $\bar{S}_0(w_L, w_R, b_L, b_R)$ is approximate the source term Thus ends the presentation of the Godunov type schemes with the source term. In order to complete the complete derivation of the schema, we have to characterize them.

3. Lyapunov exponential stability for the L^2-norm of the Shallow Water equations in trapezoidal channel

For a trapezoidal section, the area is given by

$$A = bH + mH^2, \tag{3.1}$$

where $b = b(x)$ is the channel width at the bottom and $m = m(x)$ is the inverse slope of the channel walls. Given that $H > 0$ and $m \pm 0$, we deduce the following relations from (3.1)

$$H(A,x) = \frac{\sqrt{b^2 + 4mA} - b}{2m} \tag{3.2}$$

Using (3.2), the system (1.1) can now be written as

$$\begin{cases} \partial_t A + \partial_x AV = 0, \\ \\ \partial_t V + V\partial_x V + \dfrac{g}{\sqrt{b^2+4mA}}\partial_x A = g(S_0 - S_f), \end{cases} \tag{3.3}$$

3.1. **Steady-state and linearisation.** A steady-state is a constant state A^*, V^* which satisfies the relation

$$\begin{cases} A^* V^* = Q^*, \\ \\ V^*\partial_x V^* + \dfrac{g}{\sqrt{b^{*2} + 4m^* A^*}}\partial_x A^* = g(S_0 - k\dfrac{V^{*2}}{A^*}), \end{cases} \tag{3.4}$$

where $Q^* \geq 0$ is any given constant set point and corresponds to the flow rate.

For the linearization of the system, we define the perturbation functions a and v as

$$a(t,x) = A(t,x) - A^*(x), \quad \text{and} \quad v(t,x) = V(t,x) - V^*(x). \tag{3.5}$$

Using (3.5) in (3.3) and taking into account the relations (3.4) and (3.2) we obtain the linearization of the system (2.1) around the steady state as follows:

$$\begin{pmatrix} a \\ v \end{pmatrix}_t + \begin{pmatrix} V^* & A^* \\ \dfrac{g}{\sqrt{b^{*2} + 4m^*A^*}} & V^* \end{pmatrix} \begin{pmatrix} a \\ v \end{pmatrix}_x + \begin{pmatrix} V_x^* & A_x^* \\ f_w^* & V_x^* + 2gk\dfrac{V^*}{A^*} \end{pmatrix} \begin{pmatrix} a \\ v \end{pmatrix} = 0, \tag{3.6}$$

where f_w^* is defined by

$$f_w^* = gk\frac{V^{*2}}{A^{*2}} + \frac{2m^*gA^*V^{*2}(S_0 - k\dfrac{V^{*2}}{A^*})}{\left(V^{*2}\sqrt{b^{*2} + 4m^*A^*} - gA^*\right)\left(b^{*2} + 4m^*A^*\right)}, \tag{3.7}$$

3.2. **Exponential stability of the linearized system.** In this section, we study the exponential stability of the linearized system (3.6) about a steady-state $(A^*, V^*)^T$ for the L^2-norm.

We assume that both ends of the channel are equipped with hydraulic controls (gates, pumps, mobile spillways, etc.) that allow to assign the values of the flow-rate. On-line measurements of the water levels at both ends $h(t,0)$ and $h(t,L)$ are assumed to be available for feedback control since the cross-sectional area of the water in the channel depends on the water depth. Obviously, instead of the flow-rates, we may as well consider the velocities $v(t,0)$ and $v(t,L)$ as being the control actions. Therefore we introduce the following boundary conditions: the following boundary conditions:

$$v(t,0) = k_0 a(t,0), \quad v(t,L) = k_1 a(t,L), \tag{3.8}$$

Conditions (3.8) are linear feedback static control laws with the tuning parameters k_0 and k_1.

The initial condition is given as follows

$$a(0,x) = a_0(x), \quad v(0,x) = v_0(x), \tag{3.9}$$

where $(a_0, v_0)^T \in L^2((0,L);\mathbb{R}^2)$. The Cauchy problem (3.6), (3.8) and (3.9) is well-posed. Note that the exponential stability of the linearized system is now a problem of null-stabilization for a and v.
The main result we establish is the following.

Theorem 3.1. *The linear Saint-Venant system in trapezoidal channel* (3.6), (3.8) *and* (3.9) *is exponentially stable for the L^2-norm provided that the boundary conditions satisfy*

$$k_0 \in \left(\frac{-g}{V^*(0)\sqrt{b^{*2}(0) + 4m^*(0)A^*(0)}}, -\frac{V^*(0)}{A^*(0)}\right), \tag{3.10}$$

and

$$k_1 \in \mathbb{R} \diagdown \left[\frac{-g}{V^*(L)\sqrt{b^{*2}(L) + 4m^*(L)A^*(L)}}, -\frac{V^*(L)}{A^*(L)}\right]. \tag{3.11}$$

In order to prove Theorem 1, we use a direct Lyapunov approach where the time derivative of the Lyapunov function can be madevstrictly negative definite by an appropriate choice of thevboundary conditions.